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Magnon bound states and the AdS/CFT correspondence

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Abstract

We study the spectrum of asymptotic states in the spin-chain description of planar $\mathcal{N} = 4$ SUSY Yang–Mills. In addition to elementary magnons, the asymptotic spectrum includes an infinite tower of multi-magnon bound states with an exact dispersion relation

$$\Delta - J_1 = \sqrt{Q^2 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)},$$

where the positive integer Q is the number of constituent magnons. These states account precisely for the known poles in the exact S-matrix. Like the elementary magnon, they transform in small representations of supersymmetry and are present for all values of the 't Hooft coupling. At strong coupling we identify the dual states in semiclassical string theory.

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The AdS/CFT correspondence relates the spectrum of free string theory on $\mathrm{AdS}_5 \times S^5$ to the spectrum of operator dimensions in planar $\mathcal{N} = 4$ SUSY Yang–Mills. Determining this spectrum is an interesting open problem. Starting from the gauge theory side, the problem has an elegant reformulation in terms of an integrable spin chain [1, 2] which is diagonalized by the Bethe ansatz (for reviews see [5]). Similar integrable structures have also been found in the semiclassical limit of the dual string theory [3] (see also [4]). In recent work, Hofman and Maldacena (HM) [6] have emphasized the importance of a particular limit where the spectrum simplifies on both sides of the correspondence (for earlier work on this limit see [7, 8]). In this limit both the spin chain and the dual string effectively become very long. The dynamics can then be analysed in terms of asymptotic states and their scattering.

Like the plane-wave limit [9], the HM limit focuses on operators of large R charge J_1 and scaling dimension Δ with the difference $\Delta - J_1$ held fixed. The new feature is that the $J_1, \Delta \to \infty$ limit is taken with the 't Hooft coupling $\lambda = g^2 N$ held fixed. On both sides of the correspondence the HM limit is characterized by an $SU(2|2) \times SU(2|2)$ supersymmetry with a novel central extension [13] whose generators act linearly on the worldsheet/spin-chain excitations. The spectrum consists of elementary excitations known as magnons which propagate with conserved momentum p on the infinite chain. These states are in short representations of the supersymmetry which essentially determines their dispersion relation as[1] [10–13] (see also [14])

$$\Delta - J_1 = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)}. \tag{1}$$

The magnon multiplets undergo dispersionless two-body scattering with an S-matrix which is also uniquely determined by the $SU(2|2) \times SU(2|2)$ supersymmetry up to an overall phase [13].

In any scattering theory an important possibility is that elementary excitations can form bound states. Each such object is a new asymptotic state of the theory with its own dispersion relation and S-matrix. Indeed the complete spectrum of the theory in the HM limit simply consists of all possible free multiparticle states including arbitrary numbers of each species of bound state. In this paper we will identify an infinite tower of bound states in the $SU(2)$ subsector of the theory. States in this sector are characterized by a second conserved global charge J_2 under which the elementary magnon has charge one. Our proposal is that the full spectrum in the HM limit includes a Q-magnon bound state, for each positive integer Q, with charge $J_2 = Q$ and an exact dispersion relation

$$\Delta - J_1 = \sqrt{Q^2 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)}. \tag{2}$$

In the context of the full theory we believe that these states should give rise to similar small representations of supersymmetry as the elementary magnon. In this context magnon boundstates must form complete multiplets the $SO(4)$ subgroup of the R-symmetry preserved by the groundstate of the spin chain. The charge J_2 corresponds to one of the Cartan generators of this group. Unlike the bound states discussed in [6], they are absolutely stable (for non-zero momentum) and exist for all values of the 't Hooft coupling. The occurrence of an infinite tower of BPS bound states is reminiscent of many related phenomena in string theory and supersymmetric field theory.

In the rest of the paper we will present evidence for the existence of these states both in gauge theory and in string theory. For $\lambda \ll 1$, our proposal reproduces the well-known spectrum of the Heisenberg spin chain in its thermodynamic limit. For $\lambda \gg 1$ we will identify bound states of large Q in semiclassical string theory by taking an appropriate limit of the two-spin-folded string solution of [21, 22]. However, the most important piece of evidence is valid for all values of λ: the $Q = 2$ bound state with dispersion relation (2) accounts *precisely* for the known pole in the exact two-body S-matrix of [13, 12]. In the full theory, the pole in question has a non-trivial matrix structure and is therefore associated with the piece of the S-matrix which is uniquely determined by the supersymmetries. This fits well with the fact that the corresponding bound states are BPS and their dispersion relation is also uniquely determined by SUSY. We will also identify the singularity in the Q-body S-matrix corresponding to the Q-magnon bound state.

As in integrable relativistic field theories in two dimensions [15], it seems that the bound-state spectrum and its dispersion relation places strong constraints on scattering. Further investigation of the spectrum may be useful in resolving the remaining ambiguities in the S-matrix. In particular the non-BPS bound states discussed in [6] should appear as poles in the as yet undetermined overall phase of the S-matrix.

[1] More precisely supersymmetry would allow an arbitrary function of λ to multiply the second term in the square root. However the simple λ dependence shown reproduces all known results both at weak and strong coupling.

We begin by briefly reviewing the spin-chain description of the $\mathcal{N} = 4$ theory [1]. The $SU(2)$ sector of $\mathcal{N} = 4$ SUSY Yang–Mills consists of operators of the form

$$\mathcal{O} \sim \mathrm{Tr}\left[\Phi_1^{J_1}\Phi_2^{J_2}\right] + \cdots,$$

where Φ_1 and Φ_2 are two of the three complex adjoint scalars of the theory. The dots denote all possible orderings of the fields. Each operator in this sector is characterized by two integer-valued charges J_1 and J_2 corresponding to a $U(1) \times U(1)$ subgroup of the $SU(4)$ R-symmetry group. As usual we will focus on the planar theory obtained by taking the $N \to \infty$ limit of the $SU(N)$ theory with the 't Hooft coupling $\lambda = g^2N$ held fixed.

At one loop, the dilatation operator of the theory (in the $SU(2)$ sector) can be mapped onto the Hamiltonian of the Heisenberg spin chain [1]. The spectrum of scaling dimensions can then be determined by diagonalizing the Heisenberg Hamiltonian. More precisely, at one loop, the scaling dimension Δ of an operator is related to the energy E of the corresponding eigenstate of the spin chain as

$$\Delta = L + \frac{\lambda}{8\pi^2}E. \tag{3}$$

States of the spin chain with charges J_1 and J_2 have J_2 flipped spins in a periodic chain of length $L = J_1 + J_2$. Eigenstates with a single flipped spin are known as magnons. Magnons have conserved energy ε and momentum p related by the dispersion relation

$$\varepsilon(p) = 4\sin^2\left(\frac{p}{2}\right). \tag{4}$$

Eigenstates in the sector with M flipped spins are formed as linear superpositions of M magnons. They are characterized by M individually conserved momenta p_k for $k = 1, \ldots, M$ and have total energy

$$E = \sum_{k=1}^{M}\varepsilon(p_k) = \sum_{k=1}^{M} 4\sin^2\left(\frac{p_k}{2}\right). \tag{5}$$

The problem of finding the energy levels is then reduced to determining the allowed values of the momenta p_k. These are determined by the Bethe ansatz equations

$$\exp(\mathrm{i}Lp_k) = \prod_{j\neq k}\mathcal{S}(p_k, p_j), \quad \sum_{k=1}^{M} p_k = 0 \tag{6}$$

for $k = 1, \ldots, m$. Here $\mathcal{S}$ is the two-particle S-matrix which is given as

$$\mathcal{S}(p_k, p_j) = \frac{\varphi(p_k) - \varphi(p_j) + \mathrm{i}}{\varphi(p_k) - \varphi(p_j) - \mathrm{i}} \tag{7}$$

in terms of the phase function $\varphi(p) = \cot(p/2)/2$.

Following [6], we now take the limit $L \to \infty$ with M fixed, where we also hold fixed the momenta p_k of individual magnons. This is just the standard thermodynamic limit of the spin chain (see e.g. [16, 17]). As above we will refer to this as the HM limit. It is to be contrasted both with the plane-wave or BMN limit and the limits appropriate for studying spinning strings where the momenta p_k go to zero with p_kL fixed.

The key feature of the HM limit is that, as the chain becomes very long, the magnons become dilute. Thus individual magnons propagate over many sites of the chain between interactions and can be thought of as asymptotic states. The asymptotic states undergo dispersionless two-body scattering with the S-matrix defined above. In general we may expect that as well as undergoing scattering, magnons can form bound states. Roughly speaking, a Q-magnon bound state corresponds to a state of the spin chain with Q flipped spins where the wavefunction is strongly peaked on configurations where all the flipped spins are nearly adjacent in the chain. In the thermodynamic limit where the chain length becomes infinite this notion becomes more precise and a bound state can be defined by demanding a normalizable wavefunction in the usual way. Each bound state should then be included as a new asymptotic state of the scattering theory with its own S-matrix and dispersion law.

For the Heisenberg spin chain, the spectrum of magnon bound states in the thermodynamic limit is well known (see section 5 of [17]). There is a single bound state of Q magnons, for each positive integer $Q \leqslant L/2$ with the dispersion relation

$$\varepsilon_Q(p) = \frac{4}{Q}\sin^2\left(\frac{p}{2}\right). \tag{8}$$

As we are taking an $L \to \infty$ limit, this is effectively an infinite tower.

The recipe for finding these bound states is very simple: two-magnon bound states correspond to poles in the two-body S-matrix (7) [17]. In particular, we find such a pole in $S(p_1, p_2)$ when

$$\varphi(p_1) - \varphi(p_2) = \frac{1}{2}\cot\left(\frac{p_1}{2}\right) - \frac{1}{2}\cot\left(\frac{p_2}{2}\right) = \mathrm{i} \tag{9}$$

which corresponds to a bound state[2] with $U(1)$ charge $J_2 = Q = 2$ and momentum $p = p_1 + p_2$. We solve these conditions by setting [16, 17]

$$p_1 = \frac{p}{2} + \mathrm{i}v \qquad p_2 = \frac{p}{2} - \mathrm{i}v$$

in (9) which yields $\cos(p/2) = \exp(v)$. This yields a state with energy

$$E = \varepsilon(p_1) + \varepsilon(p_2) = 4\sin^2\left(\frac{p}{4} + \mathrm{i}\frac{v}{2}\right) + 4\sin^2\left(\frac{p}{4} - \mathrm{i}\frac{v}{2}\right) = 2\sin^2\left(\frac{p}{2}\right) = \varepsilon_2(p). \tag{10}$$

Thus the position of the pole uniquely fixes the dispersion relation of the bound state.

The existence of the higher bound states with $Q > 2$, and their dispersion relation (8), can be inferred from singularities in the multi-particle S-matrix. For any integrable spin chain, this is given by a product of two-body factors. The corresponding pole appears when the momenta of the Q constituent magnons satisfy [17, 18]

$$\varphi(p_j) - \varphi(p_{j+1}) = \mathrm{i} \tag{11}$$

for $j = 1, 2, \ldots, Q - 1$. This condition is easily solved and leads directly to the bound-state dispersion relation (8).

So far we have only discussed the spectrum of the theory at one loop and only in the $SU(2)$ sector. However, assuming integrability and the spin-chain description persists in the full quantum theory, supersymmetry yields powerful constraints on the magnon dispersion relation and the two-body S-matrix [13]. These constraints provide confirmation for an earlier proposal [19] for an exact Bethe ansatz in the $SU(2)$ sector. As before the energy of an M-magnon state is the sum of the energies of individual magnons. However, the new ansatz incorporates the exact magnon dispersion relation

$$\varepsilon(p) = \frac{8\pi^2}{\lambda}\left[\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)} - 1\right] \tag{12}$$

which, because of (3), is equivalent to (1). The two-body S-matrix which enters in the Bethe ansatz equations (6) now takes the form

$$\mathcal{S}(p_k, p_j) = \frac{\varphi(p_k) - \varphi(p_j) + \mathrm{i}}{\varphi(p_k) - \varphi(p_j) - \mathrm{i}} \times \mathcal{S}_D(p_k, p_j), \tag{13}$$

[2] This state was also discussed briefly in [6].

where the phase function $\varphi(p)$ is now corrected to

$$\varphi(p) = \frac{1}{2}\cot\left(\frac{p}{2}\right)\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)}. \tag{14}$$

The first factor in (13) originates in the all-loop gauge theory ansatz of [10]. It also appears in the $SU(2)$ subsector of the full $SU(2|2)\times SU(2|2)$ S-matrix[3] of [11–13]. In contrast, $\mathcal{S}_D$ is a 'dressing factor' which is related to the undetermined overall phase of the full S-matrix. A formula for $\mathcal{S}_D$ was conjectured in [19] which passes many non-trivial tests but we will not need this here. The only fact we will use is that the dressing factor does not cancel the S-matrix pole which appears in (13) when $\varphi(p_k)-\varphi(p_j)=\mathrm{i}$.

An obvious question is what happens to the Q-magnon bound states and their dispersion law (8) described above when we move away from weak coupling. Our proposal is that they survive for all values of the coupling and have the exact dispersion relation

$$\varepsilon_Q(p) = \frac{8\pi^2}{\lambda}\left[\sqrt{Q^2+\frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)}-Q\right] \tag{15}$$

which is equivalent to (2). This formula clearly reduces to the dispersion relation (8) of the Heisenberg spin chain at weak coupling. Setting $Q=1$ we obtain the exact magnon dispersion relation (12). For $Q=2$, the proposed bound state should correspond to the pole in the exact two-body S-matrix (13). Indeed, as above, the pole position should determine the dispersion relation exactly. We will now verify this explicitly.

For magnon momenta p_1 and p_2 the new pole condition reads

$$\frac{1}{2}\cot\left(\frac{p_1}{2}\right)\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\left(\frac{p_1}{2}\right)}-\frac{1}{2}\cot\left(\frac{p_2}{2}\right)\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\left(\frac{p_2}{2}\right)}=\mathrm{i} \tag{16}$$

as before we set

$$p_1=\frac{p}{2}+\mathrm{i}v \qquad p_2=\frac{p}{2}-\mathrm{i}v$$

and solve for the bound-state momentum $p=p_1+p_2$ as a function of v. After some computation we obtain a sixth-order polynomial equation, $P_6(t)=0$, in $t=\cos(p/2)$ with coefficients polynomial in $\exp(v)$ and $a=\lambda/4\pi^2$. The polynomial $P_6(t)$ can be factored exactly into the product of a quadratic $P_2(t)$ and a quartic $P_4(t)$ which are conveniently given as

$$\begin{aligned}P_2(t) &= a(\mathrm{e}^{2v}-1)^2(1+\mathrm{e}^{2v}-2\,\mathrm{e}^{v}t)^2-4\,\mathrm{e}^{2v}(1+6\,\mathrm{e}^{2v}+\mathrm{e}^{4v}-4\,\mathrm{e}^{v}t-4\,\mathrm{e}^{3v}t)\\ P_4(t) &= a(1+\mathrm{e}^{2v}-2\,\mathrm{e}^{v}t)^2(t^2-1)+4\,\mathrm{e}^{v}(t+\mathrm{e}^{2v}t-\mathrm{e}^{v}(1+t^2)).\end{aligned} \tag{17}$$

The physical root is singled out by its weak-coupling behaviour $t=\exp(v)$ needed for agreement with the corresponding formula for the Heisenberg spin chain discussed above. Taking the limit $a\to 0$, one may easily check that the physical root belongs to the quartic equation $P_4(t)=0$ rather than the quadratic.

The next step is to extract the physical root of the quartic $P_4(t)=0$, use it to eliminate v in the energy formula

$$\begin{aligned}\varepsilon_2(p) &= \varepsilon(p_1)+\varepsilon(p_2)\\ &= \frac{8\pi^2}{\lambda}\left[\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{4}+\mathrm{i}\frac{v}{2}\right)}+\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{4}-\mathrm{i}\frac{v}{2}\right)}-2\right]\end{aligned}$$

[3] In fact the S-matrix factors corresponding to the two $SU(2|2)$ subgroups each have a single pole leading to a double pole in their product (see equation (40) of [8]). Thus, the undetermined overall phase factor should have a simple zero to obtain the expected simple pole in the complete S-matrix. The author thanks Juan Maldacena for clarifying this point.

and compare with the predicted dispersion relation (15) for the $Q = 2$ case. A necessary and sufficient condition for agreement with (15) is that the physical root of the quartic should also obey the corresponding energy conservation equation

$$\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{4}+\mathrm{i}\frac{v}{2}\right)}+\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{4}-\mathrm{i}\frac{v}{2}\right)}=\sqrt{4+\frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)}. \tag{18}$$

Squaring this equation twice and rewriting it in terms of $t = \cos(p/2)$, $\exp(v)$ and $a = \lambda/4\pi^2$ we obtain the same quartic equation $P_4(t) = 0$, with P_4 as in (17) and we are done. As for the Heisenberg spin chain, the multi-particle S-matrix has a pole corresponding to a Q-magnon bound state for each Q when condition (11) is satisfied. In principle, we could check our proposed dispersion relation (15) for $Q > 2$ by solving this condition, but we will not pursue this here.

The magnon bound states described above correspond to string theory states[4] with energy, momentum and angular momenta related as

$$\Delta - J_1 = \sqrt{J_2^2+\frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)}. \tag{19}$$

For $p = 0$, the bound state saturates the familiar BPS bound, $\Delta \geqslant J_1+J_2$, of the full $SU(2,2|4)$ superalgebra. For non-zero momentum the state appears to lie above the bound. However, as explained in [6], this is not the case because the magnon momentum can appear as a central extension of the supersymmetry algebra which modifies the BPS bound. Indeed formula (19) appears to be precisely the relevant BPS condition for all values of J_2, generalizing the single magnon result of [6]. For this reason it seems likely that all the magnon bound states discussed above give rise to similar small representations of supersymmetry as the elementary magnon.

To test the proposed spectrum of bound states further we will now go to the regime of fixed large 't Hooft coupling, $\lambda \gg 1$ and look for the corresponding states in semiclassical string theory. The HM limit is one where the energy, Δ, of the string state and one of its angular momenta J_1 both go to infinity with the difference $\Delta - J_1$ (and λ) held fixed. As in the spin chain, this thermodynamic limit is taken holding the momenta and other quantum numbers of individual world-sheet excitations fixed.

In [6] the above limit was taken for strings moving on an $R \times S^2$ subspace of $\mathrm{AdS}_5 \times S^5$ carrying a single non-zero angular momentum J_1. Classical solutions were presented corresponding to magnons of arbitrary momentum. In general the solutions correspond to folded strings with endpoints on the equator of S^2. One particularly simple case is that of momentum $p = \pm\pi$ where the dispersion relation (1) corresponds to a stationary particle on the string. A consistent state in closed string theory can be built by taking two such magnons with momenta $p = \pm\pi$. This corresponds to a special case of the folded spinning string solution of Gubser, Klebanov and Polyakov (GKP) [20]. In this limit, the string rotates around the north pole on S^2 with its endpoints moving around the equator at the speed of light. The classical energy of this state is infinite as is its angular momentum, but the difference $\Delta - J_1$ is finite and equal to $2\sqrt{\lambda}/\pi$. This matches the expected energy of the two-magnon configuration described above.

In the following we will study a simple generalization of this case with two non-zero angular momenta J_1 and J_2. Our starting point is the two-spin generalization of the GKP solution first presented in [21]. This corresponds to a string moving on an $R \times S^3$ subspace of

[4] More precisely, to obtain an allowed state of the closed string we should consider two or more excitations with total momenta equal to zero. As mentioned in [6], the central charge vanishes on such multiparticle states and so they cannot be BPS. However the exact energies of these states in the HM limit are simply the sum of the energies of their constituent magnons which are almost free in this limit.

$AdS_5 \times S^5$. String motion is described by a four-component vector $\vec{X}(\sigma,\tau) = (X_1, X_2, X_3, X_4)$ of unit length, $|\vec{X}|^2 = 1$ which specifies a point on $S^3 \subset S^5$. The additional time coordinate is eliminated with the static gauge condition $X_0 = \kappa\tau$. The relevant configuration, which corresponds to a genus-two finite gap solution of the $SU(2)$ principal chiral model, can be found using the ansatz

$$X_1 + \mathrm{i}X_2 = x_1(\sigma)\exp(\mathrm{i}\omega_1\tau) \qquad X_3 + \mathrm{i}X_4 = x_2(\sigma)\exp(\mathrm{i}\omega_2\tau)$$

with $x_1(\sigma)^2 + x_2(\sigma)^2 = 1$. The string has energy $\Delta = \sqrt{\lambda}\kappa$ and conserved angular momenta

$$J_1 = \sqrt{\lambda}\omega_1 \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} x_1(\sigma)^2 \qquad J_2 = \sqrt{\lambda}\omega_2 \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} x_2(\sigma)^2. \tag{20}$$

The solution corresponding to a folded spinning string is [21, 22]

$$x_1 = k\,\mathrm{sn}(A\sigma, k) \qquad x_2 = \mathrm{dn}(A\sigma, k) \tag{21}$$

with

$$A = \sqrt{\omega_1^2 - \omega_2^2} \qquad k = \sqrt{\frac{\kappa^2 - \omega_2^2}{\omega_1^2 - \omega_2^2}} \leqslant 1. \tag{22}$$

The Jacobian elliptic functions sn and dn are defined according to the conventions of [23]. The closed string boundary condition $\sigma \sim \sigma + 2\pi$ yields the relation

$$A = \frac{2}{\pi}K(k). \tag{23}$$

Evaluating the angular momenta on this solution we obtain

$$J_1 = \sqrt{\lambda}\omega_1\left[1 - \frac{E(k)}{K(k)}\right] \qquad J_2 = \sqrt{\lambda}\omega_2\frac{E(k)}{K(k)}, \tag{24}$$

where K and E are complete elliptic integrals.

It is convenient to introduce the variable $\rho = \omega_2/\omega_1 < 1$.

$$\begin{aligned}
\Delta &= \frac{2\sqrt{\lambda}}{\pi}\frac{\sqrt{\rho^2 + k^2(1-\rho^2)}}{\sqrt{1-\rho^2}}K(k)\\
J_1 &= \frac{2\sqrt{\lambda}}{\pi}\frac{1}{\sqrt{1-\rho^2}}(K(k) - E(k))\\
J_2 &= \frac{2\sqrt{\lambda}}{\pi}\frac{\rho}{\sqrt{1-\rho^2}}E(k).
\end{aligned} \tag{25}$$

We will now consider a limit of the HM type where $\Delta \to \infty$ and $J_1 \to \infty$ with the difference $\Delta - J_1$ held fixed. We will also hold the parameter ρ fixed. To this end we take $k \to 1$ so that

$$K(k) \simeq -\tfrac{1}{2}\log(1-k) \to \infty$$

and $E(k) \to 1$. In the limit we find the formulae

$$\Delta - J_1 = \frac{2\sqrt{\lambda}}{\pi}\frac{1}{\sqrt{1-\rho^2}} \qquad J_2 = \frac{2\sqrt{\lambda}}{\pi}\frac{\rho}{\sqrt{1-\rho^2}}. \tag{26}$$

This corresponds a one-parameter generalization of the limiting GKP solution considered in [6], the latter being the special case $\rho = 0$. Eliminating the remaining parameter ρ, we obtain the relation

$$\Delta - J_1 = 2\sqrt{\left(\frac{J_2}{2}\right)^2 + \frac{\lambda}{\pi^2}}. \tag{27}$$

As the folded string configuration is symmetric it is natural to interpret this state as consisting of two excitations each carrying half the total transverse angular momentum J_2. As before the two states have momenta $p = \pm\pi$. If we identify each of these excitations as bound states of $J_2/2$ magnons, where $J_2 \sim \sqrt{\lambda}$ then the result (27) agrees with the expected total energy calculated using the proposed dispersion relation (2).

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References

[1] Minahan J A and Zarembo K 2003 The Bethe-ansatz for $\mathcal{N} = 4$ super Yang–Mills *J. High Energy Phys.* JHEP03(2003)013 (*Preprint* hep-th/0212208)

[2] Beisert N and Staudacher M 2003 The $\mathcal{N} = 4$ SYM integrable super spin chain *Nucl. Phys.* B **670** 439 (*Preprint* hep-th/0307042)

Beisert N, Kristjansen C and Staudacher M 2003 The dilatation operator of $\mathcal{N} = 4$ super Yang–Mills theory *Nucl. Phys.* B **664** 131 (*Preprint* hep-th/0303060)

[3] Bena I, Polchinski J and Roiban R 2004 Hidden symmetries of the AdS(5)× s**5 superstring *Phys. Rev.* D **69** 046002 (*Preprint* hep-th/0305116)

[4] Mandal G, Suryanarayana N V and Wadia S R 2002 Aspects of semiclassical strings in AdS(5) *Phys. Lett.* B **543** 81 (*Preprint* hep-th/0206103)

[5] Beisert N 2005 *Phys. Rep.* **405** 1 (*Preprint* hep-th/0407277)

Beisert N 2004 *C. R. Phys.* **5** 1039 (*Preprint* hep-th/0409147)

Zarembo K 2004 *C. R. Phys.* **5** 1081

Zarembo K 2005 *Fortsch. Phys.* **53** 647 (*Preprint* hep-th/0411191)

Plefka J 2005 *Preprint* hep-th/0507136

[6] Hofman D M and Maldacena J 2006 Giant magnons *Preprint* hep-th/0604135

[7] Mann N and Polchinski J 2005 Bethe ansatz for a quantum supercoset sigma model *Phys. Rev.* D **72** 086002 (*Preprint* hep-th/0508232)

Ambjorn J, Janik R A and Kristjansen C 2006 Wrapping interactions and a new source of corrections to the spin-chain/string duality *Nucl. Phys.* B **736** 288 (*Preprint* hep-th/0510171)

Klose T and Zarembo K 2006 Bethe ansatz in stringy sigma models *Preprint* hep-th/0603039

Gromov N, Kazakov V, Sakai K and Vieira P 2006 Strings as multi-particle states of quantum sigma-models *Preprint* hep-th/0603043

Arutyunov G and Tseytlin A A 2006 On highest-energy state in the su(1|1) sector of $\mathcal{N} = 4$ super Yang–Mills *Preprint* hep-th/0603113

Arutyunov G and Frolov S 2006 On AdS(5)× S**5 string S-matrix *Preprint* hep-th/0604043

[8] Janik R A 2006 The AdS(5) × S**5 superstring worldsheet S-matrix and crossing symmetry *Preprint* hep-th/0603038

[9] Berenstein D, Maldacena J M and Nastase H 2002 Strings in flat space and pp waves from $\mathcal{N} = 4$ super Yang Mills *J. High Energy Phys.* JHEP04(2002)013 (*Preprint* hep-th/0202021)

[10] Beisert N, Dippel V and Staudacher M 2004 A novel long range spin chain and planar $\mathcal{N} = 4$ super Yang–Mills *J. High Energy Phys.* JHEP07(2004)075 (*Preprint* hep-th/0405001)

[11] Staudacher M 2004 The factorized S-matrix of CFT/AdS *J. High Energy Phys.* JHEP05(2005)054 (*Preprint* hep-th/0412188)

[12] Beisert N and Staudacher M 2005 Long-range PSU(2,2|4) Bethe ansaetze for gauge theory and strings *Nucl. Phys.* B **727** 1 (*Preprint* hep-th/0504190)

[13] Beisert N 2005 The su(2|2) dynamic S-matrix *Preprint* hep-th/0511082

[14] Santambrogio A and Zanon D 2002 Exact anomalous dimensions of $\mathcal{N} = 4$ Yang–Mills operators with large R charge *Phys. Lett.* B **545** 425 (*Preprint* hep-th/0206079)

Berenstein D, Correa D H and Vazquez S E 2006 All loop BMN state energies from matrices *J. High Energy Phys.* JHEP02(2006)048 (*Preprint* hep-th/0509015)

[15] Zamolodchikov A B and Zamolodchikov A B 1979 Factorized s-matrices in two dimensions as the exact solutions of certain relativistic quantum field models *Ann. Phys.* **120** 253

[16] Karbach M and Müller G 1998 Introduction to the Bethe ansatz I *Preprint* cond-mat/9809162
[17] Faddeev L D 1996 How algebraic Bethe ansatz works for integrable model *Preprint* hep-th/9605187
[18] Babbit D and Thomas L 1978 *J. Math. Phys.* **19** 1699
[19] Arutyunov G, Frolov S and Staudacher M 2004 Bethe ansatz for quantum strings *J. High Energy Phys.* JHEP10(2004)016 (*Preprint* hep-th/0406256)
[20] Gubser S S, Klebanov I R and Polyakov A M 2002 A semi-classical limit of the gauge/string correspondence *Nucl. Phys.* B **636** 99 (*Preprint* hep-th/0204051)
[21] Frolov S and Tseytlin A A 2003 Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors *Phys. Lett.* B **570** 96 (*Preprint* hep-th/0306143)
[22] Arutyunov G, Frolov S, Russo J and Tseytlin A A 2003 Spinning strings in AdS(5) $\times$ s**5 and integrable systems *Nucl. Phys.* B **671** 3 (*Preprint* hep-th/0307191)
[23] Gradshteyn I S and Ryzhik I M 1994 *Table of Integrals and Series* (New York: Academic)